The non-consistency of a pure Yang-Mills type formulation for gravity revisited

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Abstract

A perturbative regime based on contorsion as a dynamical variable and metric as a (classical) fixed background, is performed in the context of a pure Yang-Mills formulation based on $GL(3,R)$ gauge group. In the massless case we show that the theory propagates three degrees of freedom and only one is a non-unitary mode. Next, we introduce quadratical terms dependent on torsion, which preserve parity and general covariance. The linearized version reproduces an analogue Hilbert-Einstein-Fierz-Pauli unitary massive theory plus three massless modes, two of them non-unitary ones.

1 Introduction

There were some contributions on the exploration of classical consistency of a pure Yang-Mills (YM) type formulation for gravity, including the cosmological extension one [1, 2], among others. In those references, Einstein theory is recovered after the imposition of torsion constraints.

Unfortunately, the path to a quantum version (if it is finally possible) is not straightforward. For example, it is well known that the Lagrangian of a pure YM theory based on the Lorentz group $SO(3,1) \simeq SL(2,C)$ [3] leads to a non-positive Hamiltonian (due to non-compactness of the aforementioned gauge group) and, then

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the canonical quantization procedure fails. However, there is a possible way out if it is considered an extension of the YM model thinking about a theory like Gauss-Bonnet with Torsion (GBT)[3] and, moreover there exists a possible family of GBT type theories from which it can be recovered unitarity[4].

The aim of this work is to expose with some detail a similar (and obvious) situation about non-unitarity in a YM formulation with $GL(3, R)$ as a gauge group in both massless and massive theories. Particularly, the massive version considered here is a parity presering quadratical set of terms which depends on torsion (the old idea about considering T^2-terms in a dynamical theory of torsion has been considered in the past[5]) and, at a perturbative regime gives rise to a Fierz-Pauli massive term. Throughout this work we follow the spirit of Kibble's idea [6] treating the metric as a fixed background meanwhile the torsion (contorsion) shall be considered as a dynamical field and it would be thought as a quantum fluctuation around a classical fixed background.

This paper is organized as follows. The next section is devoted to a brief review on notation of the cosmologically extended YM formulation[1] in N-dimensions and its topologically massive version in $2 + 1$[2]. In section 3, we consider the scheme of linearization of the massless theory around a fixed Minkowskian background, allowing fluctuations on torsion. Next, the Lagrangian analysis of constraints and construction of the reduced action is performed, showing that this theory does propagate degrees of freedom, including a ghost. In section 4, we introduce an appropiate T^2-term, which preserves parity, general covariance and its linearization gives rise to a Fierz-Pauli mass term. There, the non-positive definite Hamiltonian problem get worse: the Lagrangian analisys shows that the theory have more non-unitary degrees of freedom and we can't expect other thing. Gauge transformations are explored in section 5. Although T^2-terms provide mass only to some spin component of contorsion, the linearized theory loses the gauge invariance and there is no any residual invariance. This is clearly established through a standard procedure for the study of possible chains of gauge generators[7]. We end up with some concluding remarks.

2 A pure Yang-Mills formulation for gravity

Let M be a N-dimensional manifold with a metric, $g_{\mu\nu}$ provided. A (principal) fiber bundle is constructed with M and a 1-form connection is given, $(A_\lambda)^\mu{}_\nu$ which will be non metric dependent. The affine connection transforms as $A_\lambda{}' = UA_\lambda U^{-1} + U\partial_\lambda U^{-1}$ under $U \in GL(N,R)$. Torsion and curvature tensors are $T^\mu{}_{\lambda\nu} = (A_\lambda)^\mu{}_\nu - (A_\nu)^\mu{}_\lambda$ and $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$, respectively. Components of the Riemann tensor are $R^\sigma{}_{\alpha\mu\nu} \equiv (F_{\nu\mu})^\sigma{}_\alpha$. The gauge invariant action with cosmological contribution is[1]

$$S^{(N)}{}_0 = \kappa^{2(4-N)} \Big\langle -\frac{1}{4}\, tr\, F^{\alpha\beta} F_{\alpha\beta} + q(N)\lambda^2 \Big\rangle \ , \tag{1}$$

where κ^2 is in length units, $\langle ... \rangle \equiv \int d^N x \sqrt{-g}(...)$, λ is the cosmologic constant and the parameter $q(N) = 2(4-N)/(N-2)^2(N-1)$ depends on dimension. The shape of $q(N)$ allows the recovering of (free) Einstein's equations as a particular solution when the torsionless Lagrangian constraints are imposed and it changes it sign when $N > 5$. The field equations are $\frac{1}{\sqrt{-g}}\, \partial_\alpha(\sqrt{-g}\ F^{\alpha\lambda}) + [A_\alpha, F^{\alpha\lambda}] = 0$ and $T_g{}^{\alpha\beta} = -\kappa^2 g^{\alpha\beta} \lambda^2$ where $T_g{}^{\alpha\beta} \equiv \kappa^2\, tr[F^{\alpha\sigma} F^\beta{}_\sigma - \frac{g^{\alpha\beta}}{4}\, F^{\mu\nu} F_{\mu\nu}]$ is the energy-momentum tensor of gravity.

Particularly, in $2+1$ dimension take place the topologically massive version[2], this means

$$S^{(3)}{}_{tm} = S^{(3)L}{}_0 + \frac{m\kappa^2}{2} \Big\langle \varepsilon^{\mu\nu\lambda}\, tr\big(A_\mu \partial_\nu A_\lambda + \frac{2}{3}\, A_\mu A_\nu A_\lambda\big) \Big\rangle \ , \tag{2}$$

which contains the cosmologically extended TMG[8] when the torsionless constraints are introduced through a suitable set of Lagrangian multipliers. Obviously, (2) does not preserve parity.

3 Linearization of the massless theory

With a view on the performing of a perturbative study of the massive model, we wish to note some aspects of the variational analysis of free action (1) in $2+1$ dimensions.

As we had said above, the connection shall be considered as a dynamical field whereas the space-time metric would be a fixed background, in order to explore (in some sense) the isolated behavior of torsion (contorsion) and avoid higher order terms in the field equations. For simplicity we shall assume $\lambda = 0$.

Then, let us consider a Minkowskian space-time with a metric $diag(-1, 1, 1)$ provided and, obviously with no curvature nor torsion. The notation is

$$\overline{g}_{\alpha\beta} = \eta_{\alpha\beta} \ , \tag{3}$$

$$\overline{F}^{\alpha\beta} = 0 \ , \tag{4}$$

$$\overline{T}^{\lambda}{}_{\mu\nu} = 0 \ . \tag{5}$$

It can be observed that curvature $\overline{F}^{\alpha\beta} = 0$ and torsion $\overline{T}^{\lambda}{}_{\mu\nu} = 0$, in a space-time with metric $\overline{g}_{\alpha\beta} = \eta_{\alpha\beta}$ satisfy the background equations, $\frac{1}{\sqrt{-\overline{g}}}\ \partial_\alpha(\sqrt{-\overline{g}}\ \overline{F}^{\alpha\lambda}) + [\overline{A}_\alpha, \overline{F}^{\alpha\lambda}] = 0$ and $\overline{T}_g{}^{\alpha\beta} = 0$, identically.

Thinking in variations

$$A_\mu = \overline{A}_\mu + a_\mu \quad , \quad |a_\mu| \ll 1 \ , \tag{6}$$

for this case $\overline{A}_\mu = 0$. Then, action (1) takes the form

$$S^{(3)L}{}_0 = \kappa^2 \Big\langle -\frac{1}{4}\, tr\, f^{\alpha\beta}(a) f_{\alpha\beta}(a) \Big\rangle \ , \tag{7}$$

where $f_{\alpha\beta}(a) = \partial_\alpha a_\beta - \partial_\beta a_\alpha$ and (7) is gauge invariant under

$$\delta a_\mu = \partial_\mu \omega \ , \tag{8}$$

with the gauge group $G = U(1) \times ...3^2... \times U(1)$.

Let us suppose we take a *Weitzenböck* space-time instead a Minkowski one, as the fixed background. Then, the condition (5) must be relaxed (i.e.: $\overline{T}^{\lambda}{}_{\mu\nu} \neq 0$) and the linearized action would be $S^{(3)L}_{Weitzenböck} = S^{(3)L}{}_0 - \kappa^2 \big\langle tr f^{\alpha\beta}(a)[\overline{A}_\alpha, a_\beta] + \frac{1}{2}\, tr\, [\overline{A}^\alpha, a^\beta]\big([\overline{A}_\alpha, a_\beta] - [\overline{A}_\beta, a_\alpha]\big)\big\rangle$, which now is gauge variant under (8). In this context, the gauge invariance can be recovered through a similar technique of *Stückelberg*

auxilliary fields for a *Proca* type model. This situation suggests we would be in presence of possible massive contributions (surely with ghosts) due just to the background.

In order to describe in detail the action (7), let us consider the following decomposition for perturbed connection

$$(a_\mu)^\alpha{}_\beta = \epsilon^{\sigma\alpha}{}_\beta k_{\mu\sigma} + \delta^\alpha{}_\mu v_\beta - \eta_{\mu\beta} v^\alpha \ , \tag{9}$$

where $k_{\mu\nu} = k_{\nu\mu}$ and v_μ are the symmetric and antisymmetric parts of the rank two perturbed contorsion (i. e., the rank two contorsion is $K_{\mu\nu} \equiv -\frac{1}{2}\epsilon^{\sigma\rho}{}_\nu K_{\sigma\mu\rho}$), respectively. It can be noted that decomposition (9) is not were performed in irreducible spin components and explicit writing down of the traceless part of $k_{\mu\nu}$ would be needed. This component will be considered when the study of reduced action be performed. Using (9) in (7), we get

$$S^{(3)L}{}_0 = \kappa^2 \left\langle k_{\mu\nu} \Box k^{\mu\nu} + \partial_\mu k^{\mu\sigma} \partial_\nu k^\nu{}_\sigma - 2\epsilon^{\sigma\alpha\beta} \partial_\alpha v_\beta \partial_\nu k^\nu{}_\sigma - v_\mu \Box v^\mu + (\partial_\mu v^\mu)^2 \right\rangle \ , \tag{10}$$

which is gauge invariant under the following transformation rules (induced by (8))

$$\delta k_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu \ , \tag{11}$$

$$\delta v_\mu = -\epsilon^{\sigma\rho}{}_\mu \partial_\sigma \xi_\rho \ , \tag{12}$$

with $\xi_\mu \equiv \frac{1}{4}\epsilon^\beta{}_{\alpha\mu} w^\alpha{}_\beta$. These transformation rules clearly show that only the antisymmetric part of w is needed (i. e.: only three gauge fixation would be chosen).

In expression (10) we can observe that the term $v_\mu \Box v^\mu$ have a wrong sign, telling us about the non-unitarity property of the theory. However, field equations are

$$2\Box k_{\mu\nu} - \partial_\mu \partial_\sigma k^\sigma{}_\nu - \partial_\nu \partial_\sigma k^\sigma{}_\mu + \epsilon^{\sigma\rho}{}_\mu \partial_\nu \partial_\sigma v_\rho + \epsilon^{\sigma\rho}{}_\nu \partial_\mu \partial_\sigma v_\rho = 0 \ , \tag{13}$$

$$\epsilon^{\sigma\rho\beta} \partial_\sigma \partial_\mu k^\mu{}_\rho + \Box v^\beta + \partial^\beta \partial_\mu v^\mu = 0 \ , \tag{14}$$

and note that (13) satisfies the consistence condition

$$\Box k - \partial_\mu \partial_\nu k^{\mu\nu} = 0 \ . \tag{15}$$

Divergence of (14) says that $\partial_\mu v^\mu$ is a massless 0-form then, if we define $\hat{\partial}_\sigma \equiv \Box^{-\frac{1}{2}}\partial_\sigma$, the following relation can be written

$$v^\beta = -\epsilon^{\sigma\rho\beta}\hat{\partial}_\sigma\hat{\partial}_\mu k^\mu{}_\rho \ , \tag{16}$$

up to a massless-transverse 1-form. Using (16) in (13), gives rise to

$$\Box k_{\mu\nu} - \partial_\mu\partial_\sigma k^\sigma{}_\nu - \partial_\nu\partial_\sigma k^\sigma{}_\mu + \partial_\mu\partial_\nu k = 0 \ , \tag{17}$$

up to a massless 0-form. This last equation with condition (15) would suggests a possible equivalence with the model for gravitons of the linearized Hilbert-Einstein theory (i. e.: free gravity in $2+1$ does not propagate degrees of freedom). However, this suggestion is wrong because we were drop-out too many light (harmonics) modes and, then it is necessary to take in to account both massive and massless complete sets of modes.

First at all, let us to study the system of Lagrangian constraints in order to explore the number of degrees of freedom. A possible approach consists in a $2+1$ decomposition of the action (10) in the way

$$\begin{aligned} S^{(3)L}{}_0 = \kappa^2\big\langle[-\dot{k}_{0i} + 2\partial_i k_{00} - 2\partial_n k_{ni} - 2\epsilon_{in}\dot{v}_n + 2\epsilon_{in}\partial_n v_0]\dot{k}_{0i} + \dot{k}_{ij}\dot{k}_{ij} \\ +[2\epsilon_{nj}\partial_n k_{00} + 2\epsilon_{nj}\partial_m k_{nm} - \dot{v}_j - 2\partial_j v_0]\dot{v}_j + 2(\dot{v}_0)^2 + k_{00}\Delta k_{00} \\ -2k_{0i}\Delta k_{0i} + k_{ij}\Delta k_{ij} - (\partial_i k_{i0})^2 + \partial_n k_{ni}\partial_m k_{mi} - 2\epsilon_{ij}\partial_i v_j \partial_n k_{n0} \\ -2\epsilon_{lm}\partial_m v_0 \partial_n k_{nl} + v_0\Delta v_0 - v_i\Delta v_i + (\partial_n v_n)^2\big\rangle \ , \end{aligned} \tag{18}$$

and using a Transverse-Longitudinal (TL) decomposition[9] with notation

$$k_{00} \equiv n \ , \tag{19}$$

$$h_{i0} = h_{0i} \equiv \partial_i k^L + \epsilon_{il}\partial_l k^T \ , \tag{20}$$

$$k_{ij} = k_{ji} \equiv (\eta_{ij}\Delta - \partial_i\partial_j)k^{TT} + \partial_i\partial_j k^{LL} + (\epsilon_{ik}\partial_k\partial_j + \epsilon_{jk}\partial_k\partial_i)k^{TL} \ , \tag{21}$$

$$v_0 \equiv q \ , \tag{22}$$

$$v_i \equiv \partial_i v^L + \epsilon_{il}\partial_l v^T \ , \tag{23}$$

where $\Delta \equiv \partial_i\partial_i$, we can rewrite (18) as follows

$$\begin{aligned} S^{(3)L}{}_0 = \kappa^2 \big\langle & \dot{k}^L \Delta \dot{k}^L + \dot{k}^T \Delta \dot{k}^T + \dot{v}^L \Delta \dot{v}^L + \dot{v}^T \Delta \dot{v}^T + 2\dot{v}^L \Delta \dot{k}^T - 2\dot{v}^T \Delta \dot{k}^L \\ & +(\Delta \dot{k}^{TT})^2 + (\Delta \dot{k}^{LL})^2 + 2(\Delta \dot{k}^{TL})^2 + 2(\dot{q})^2 - 2n\Delta \dot{k}^L + 2n\Delta \dot{v}^T \\ & +2q\Delta \dot{v}^L - 2q\Delta \dot{k}^T + 2\Delta k^{LL}\Delta \dot{k}^L + 2\Delta k^{TL}\Delta \dot{k}^T + 2\Delta k^{LL}\Delta \dot{v}^T \\ & -2\Delta k^{TL}\Delta \dot{v}^L + q\Delta q + n\Delta n + (\Delta k^L)^2 + 2(\Delta k^T)^2 + 2(\Delta v^L)^2 \\ & +(\Delta v^T)^2 + 2\Delta v^T \Delta k^L + 2q\Delta^2 k^{TL} + \Delta k^{TT}\Delta^2 k^{TT} + \Delta k^{TL}\Delta^2 k^{TL} \big\rangle \ . \end{aligned} \tag{24}$$

Primary Lagrangian constraints, joined to some relations for accelerations can be obtained through an inspection on field equations, which arise from (24). Also, as well as the set of primary constraints we need to perform a gauge fixation. A "Coulomb" gauge is defined by the constraints $\partial_i k_{i\mu} = 0$, which can be rewritten in terms of the TL-decomposition as follows (up to harmonics)

$$k^L = k^{LL} = k^{TL} = 0 \ , \tag{25}$$

and preservation provide the next relations for longitudinal velocities and accelerations

$$\dot{k}^L = \dot{k}^{LL} = \dot{k}^{TL} = 0 \ , \tag{26}$$

$$\ddot{k}^L = \ddot{k}^{LL} = \ddot{k}^{TL} = 0 \ . \tag{27}$$

Equations (25) and (26) are six Lagrangian constraints.

Field equations with the help of gauge constraints, give the following five (primary) constraints

$$n = 0 \ , \tag{28}$$

$$v^T = 0 \ , \tag{29}$$

$$\dot{v}^T = 0 \ , \tag{30}$$

$$\dot{k}^T - \dot{v}^L + q = 0 \ , \tag{31}$$

$$\Delta k^T - \Delta v^L + \dot{q} = 0 \ , \tag{32}$$

and accelerations are related through

$$\ddot{v}^T = -\dot{n} \ , \tag{33}$$

$$\ddot{k}^T + \ddot{v}^L = \dot{q} + 2\Delta k^T \ , \tag{34}$$

$$\ddot{k}^{TT} = \Delta k^{TT} \ , \tag{35}$$

$$\ddot{q} = \Delta q \ . \tag{36}$$

Systematic preservation of constraints (28) and (32) provide a new constraint

$$\dot{n} = 0 \ , \tag{37}$$

and accelerations

$$\ddot{n} = 0 \ , \tag{38}$$

$$\ddot{v}^T = 0 \ , \tag{39}$$

$$\ddot{k}^T = \Delta k^T \ , \tag{40}$$

$$\ddot{v}^L = \Delta v^L \ . \tag{41}$$

In short, there is a set of twelve constraints

$$n = \dot{n} = v^T = \dot{v}^T = k^L = \dot{k}^L = k^{LL} = \dot{k}^{LL} = k^{TL} = \dot{k}^{TL} = 0 \ , \tag{42}$$

$$\dot{k}^T - \dot{v}^L + q = 0 \ , \tag{43}$$

$$\Delta k^T - \Delta v^L + \dot{q} = 0 \ , \tag{44}$$

then, there are three degrees of freedom and constraint system give rise to reduced action

$$S^{(3)L*}{}_0 = \kappa^2 \Big\langle 4\dot{k}^T \Delta \dot{k}^T + 4(\Delta k^T)^2 + 4(\dot{q})^2 + 4q\Delta q + (\Delta \dot{k}^{TT})^2 + \Delta k^{TT} \Delta^2 k^{TT} \Big\rangle \ . \tag{45}$$

Introducing notation

$$Q \equiv 2q \ , \tag{46}$$

$$Q^T \equiv 2(-\Delta)^{\frac{1}{2}} k^T \ , \tag{47}$$

$$Q^{TT} \equiv \Delta k^{TT} \ , \tag{48}$$

the reduced action is rewritten as follows

$$S^{(3)L*}{}_0 = \kappa^2 \Big\langle Q\Box Q - Q^T \Box Q^T + Q^{TT} \Box Q^{TT} \Big\rangle \ , \tag{49}$$

showing two unitary and one non-unitary modes, then the Hamiltonian is not positive definite.

4 YM gravity with parity preserving massive term

It can be possible to write down a massive version which respect parity, for example

$$S^{(3)}{}_m = S^{(3)}{}_0 - \frac{m^2\kappa^2}{2} \Big\langle T^\sigma{}_{\sigma\nu} T^\rho{}_\rho{}^\nu - T^{\lambda\mu\nu} T_{\mu\lambda\nu} - \frac{1}{2} T^{\lambda\mu\nu} T_{\lambda\mu\nu} \Big\rangle \ . \tag{50}$$

In a general case, if we allow independent variations on metric and connection two types of field equations can be obtained. On one hand, variations on metric give rise to the expression of the gravitacional energy-momentum tensor, $T_g{}^{\alpha\beta} \equiv \kappa^2\, tr[F^{\alpha\sigma} F^\beta{}_\sigma - \frac{g^{\alpha\beta}}{4} F^{\mu\nu} F_{\mu\nu}]$, in other words

$$T_g{}^{\alpha\beta} = -T_t^{\alpha\beta} - \kappa^2 g^{\alpha\beta} \lambda^2 \ , \tag{51}$$

where $T_t^{\alpha\beta} \equiv -m^2\kappa^2[3t^{\alpha\sigma} t^\beta{}_\sigma + 3t^{\sigma\alpha} t_\sigma{}^\beta - t^{\alpha\sigma} t_\sigma{}^\beta - t^{\sigma\alpha} t^\beta{}_\sigma - (t^{\alpha\beta} + t^{\beta\alpha}) t_\sigma{}^\sigma - \frac{5g^{\alpha\beta}}{2} t^{\mu\nu} t_{\mu\nu} + \frac{3g^{\alpha\beta}}{2} t^{\mu\nu} t_{\nu\mu} + \frac{g^{\alpha\beta}}{2} (t_\sigma{}^\sigma)^2]$ is the torsion contribution to the energy-momentum distribution and $t^{\alpha\beta} \equiv \frac{\varepsilon^{\mu\nu\alpha}}{2} T^\beta{}_{\mu\nu}$. This says, for example, that the quest of possible black hole solutions must reveal a dependence on parameters m^2 and λ^2.

On the other hand, variations on connection provide the following equations

$$\frac{1}{\sqrt{-g}}\, \partial_\alpha(\sqrt{-g}\ F^{\alpha\lambda}) + [A_\alpha, F^{\alpha\lambda}] = J^\lambda \ , \tag{52}$$

where the current is $(J^\lambda)^\nu{}_\sigma = m^2(\delta^\lambda{}_\sigma K^\rho{}_\rho{}^\nu - \delta^\nu{}_\sigma K^\rho{}_\rho{}^\lambda + 2K^\nu{}_\sigma{}^\lambda)$ and the contorsion $K^\lambda{}_{\mu\nu} \equiv \frac{1}{2}(T^\lambda{}_{\mu\nu} + T_\mu{}^\lambda{}_\nu + T_\nu{}^\lambda{}_\mu)$. We can observe in (52) that contorsion and metric appear as sources of gravity, where the cosmological contribution is obviously hide in space-time metric. In a weak torsion regime, equation (52) takes a familiar shape: $D_\alpha F^{\alpha\lambda} = J^\lambda$, where D_α is the covariant derivative computed with the Christoffel's symbols.

Now we explore the perturbation of the massive case given at (50) and with the help of (9), the linearized action is

$$
\begin{aligned}
S^{(3)L}{}_m = \kappa^2\big\langle & k_{\mu\nu}\Box k^{\mu\nu} + \partial_\mu k^{\mu\sigma}\partial_\nu k^\nu{}_\sigma - 2\epsilon^{\sigma\alpha\beta}\partial_\alpha v_\beta \partial_\nu k^\nu{}_\sigma \\
& -v_\mu\Box v^\mu + (\partial_\mu v^\mu)^2 - m^2(k_{\mu\nu}k^{\mu\nu} - k^2)\big\rangle \ .
\end{aligned}
\tag{53}
$$

Using a TL-decomposition defined by (19)-(23), we can write (53) in the way

$$
\begin{aligned}
S^{(3)L}{}_m = \kappa^2\big\langle & \dot{k}^L\Delta\dot{k}^L + \dot{k}^T\Delta\dot{k}^T + \dot{v}^L\Delta\dot{v}^L + \dot{v}^T\Delta\dot{v}^T + 2\dot{v}^L\Delta\dot{k}^T - 2\dot{v}^T\Delta\dot{k}^L \\
& +(\Delta\dot{k}^{TT})^2 + (\Delta\dot{k}^{LL})^2 + 2(\Delta\dot{k}^{TL})^2 + 2(\dot{q})^2 - 2n\Delta\dot{k}^L + 2n\Delta\dot{v}^T \\
& +2q\Delta\dot{v}^L - 2q\Delta\dot{k}^T + 2\Delta k^{LL}\Delta\dot{k}^L + 2\Delta k^{TL}\Delta\dot{k}^T + 2\Delta k^{LL}\Delta\dot{v}^T \\
& -2\Delta k^{TL}\Delta\dot{v}^L + q\Delta q + n\Delta n + (\Delta k^L)^2 + 2(\Delta k^T)^2 + 2(\Delta v^L)^2 \\
& +(\Delta v^T)^2 + 2\Delta v^T\Delta k^L + 2q\Delta^2 k^{TL} + \Delta k^{TT}\Delta^2 k^{TT} + \Delta k^{TL}\Delta^2 k^{TL} \\
& +m^2[-2k^L\Delta k^L - 2k^T\Delta k^T - 2(\Delta k^{TL})^2 - 2n(\Delta k^{TT} + \Delta k^{LL}) \\
& +2\Delta k^{TT}\Delta k^{LL}]\big\rangle \ .
\end{aligned}
\tag{54}
$$

Here, there is no gauge freedom (as it shall be confirmed in next section) and field equations provide primary constraints and some accelerations. The preservation procedure give rise expressions for all accelerations

$$\ddot{n} = \Delta\dot{k}^L \ , \tag{55}$$

$$\ddot{k}^L = -\Delta\dot{k}^{TT} + \dot{n} \ , \tag{56}$$

$$\ddot{k}^T = \Delta \dot{k}^{TL} \ , \tag{57}$$

$$\ddot{k}^{LL} = \dot{k}^L + \dot{v}^T + m^2 k^{TT} - m^2 \Delta^{-1} n \ , \tag{58}$$

$$\ddot{k}^{TL} = \frac{1}{2}(\dot{k}^T - \dot{v}^L + \Delta k^{TL} + q - 2m^2 k^{TL}) \ , \tag{59}$$

$$\ddot{k}^{TT} = \Delta k^{TT} + m^2 k^{LL} - m^2 \Delta^{-1} n \ , \tag{60}$$

$$\ddot{q} = \frac{1}{2}(\Delta \dot{v}^L - \Delta \dot{k}^T + \Delta^2 k^{TL} + \Delta q) \ , \tag{61}$$

$$\ddot{v}^L = -\dot{q} + 2\Delta v^L \ , \tag{62}$$

$$\ddot{v}^T = -\Delta \dot{k}^{TT} - \Delta \dot{k}^{LL} + \Delta k^L + \Delta v^T \ , \tag{63}$$

and a set of eight constraints

$$\dot{v}^T - \dot{k}^L + n - m^2(k^{TT} + k^{LL}) = 0 \ , \tag{64}$$

$$\Delta \dot{k}^{LL} - \Delta k^L - \Delta v^T + m^2 k^L = 0 \ , \tag{65}$$

$$\Delta \dot{k}^{TL} - \dot{q} - \Delta k^T + \Delta v^L + m^2 k^T = 0 \ , \tag{66}$$

$$\Delta \dot{k}^{LL} - \Delta k^L - \Delta v^T + m^2(\dot{k}^{TT} + \dot{k}^{LL}) = 0 \ , \tag{67}$$

$$\dot{k}^L + \Delta k^{TT} - n = 0 \ , \tag{68}$$

$$\dot{k}^T - \Delta k^{TL} = 0 \ , \tag{69}$$

$$\dot{v}^T + \Delta k^{TT} + m^2(k^{TT} + k^{LL}) - 2m^2 \Delta^{-1} n = 0 \ , \tag{70}$$

$$\dot{n} - \Delta k^L = 0 \ , \tag{71}$$

which says that this massive theory propagates five degrees of freedom. In order to explore the physical content, we can take a short path to this purpose and it means to start with a typical transverse-traceless (Tt) decomposition instead the TL-decomposition one. Notation for the Tt-decomposition of fields is

$$k_{\mu\nu} = k^{Tt}{}_{\mu\nu} + \hat{\partial}_\mu \theta^T{}_\nu + \hat{\partial}_\nu \theta^T{}_\mu + \hat{\partial}_\mu \hat{\partial}_\nu \psi + \eta_{\mu\nu} \phi \ , \tag{72}$$

$$v_\mu = v^T{}_\mu + \hat{\partial}_\mu v \ , \tag{73}$$

with the subsidiary conditions

$$k^{Tt\mu}{}_{\mu} = 0 \quad , \quad \partial^\mu k^{Tt}{}_{\mu\nu} = 0 \quad , \quad \partial^\mu \theta^T{}_\mu = 0 \quad , \quad \partial^\mu v^T{}_\mu = 0 \ . \tag{74}$$

Action (53) is

$$S^{(3)L}{}_m = \kappa^2 \big\langle k^{Tt}{}_{\mu\nu}(\Box - m^2)k^{Tt\mu\nu} - \theta^T{}_\mu(\Box - 2m^2)\theta^T{}_\mu - 2\epsilon^{\sigma\alpha\beta}\partial_\alpha v^T{}_\beta \Box^{\frac{1}{2}}\theta^T{}_\sigma$$
$$-v^T{}_\mu \Box v^{T\mu} + 2v\Box v + 2\phi\Box\phi + 4m^2\psi\phi + 6m^2\phi^2 \big\rangle \ . \tag{75}$$

A new transverse variable, $a^T{}_\mu$ is introduced through

$$\theta^T{}_\mu \equiv \epsilon_\mu{}^{\alpha\beta}\hat{\partial}_\alpha a^T{}_\beta \ , \tag{76}$$

and the action (75) is rewritten as

$$S^{(3)L}{}_m = \kappa^2 \big\langle k^{Tt}{}_{\mu\nu}(\Box - m^2)k^{Tt\mu\nu} - a^T{}_\mu(\Box - 2m^2)a^T{}_\mu - 2a^T{}_\mu \Box v^{T\mu}$$
$$-v^T{}_\mu \Box v^{T\mu} + 2v\Box v + 2\phi\Box\phi + 4m^2\psi\phi + 6m^2\phi^2 \big\rangle \ . \tag{77}$$

The field equations are

$$(\Box - m^2)k^{Tt}{}_{\mu\nu} = 0 \ , \tag{78}$$

$$\Box v^T{}_\mu = 0 \ , \tag{79}$$

$$\Box v = 0 \ , \tag{80}$$

$$a^T{}_\mu = 0 \ , \tag{81}$$

$$\psi = \phi = 0 \ , \tag{82}$$

and reduced action is

$$S^{(3)L*}{}_m = \kappa^2 \big\langle k^{Tt}{}_{\mu\nu}(\Box - m^2)k^{Tt\mu\nu} + 2v\Box v - v^T{}_\mu \Box v^{T\mu} \big\rangle \ , \tag{83}$$

saying that the contorsion propagates two massive helicities ± 2, one massless spin-0 and two massless ghost vectors. Then, there is not positive definite Hamiltonian. This observation can be confirmed in the next section when we shall write down the Hamiltonian density and a wrong sign appears in the kinetic part corresponding to the canonical momentum of v_i (see eq. (95)).

5 Gauge transformations

The quadratical Lagrangian density dependent in torsion and presented in (50), has been constructed without free parameters, with the exception of m^2, of course. It has a particular shape which only gives mass to the spin 2 component of the contorsion, as we see in the perturbative regime. Let us to comment about de non existence of any possible "residual" gauge invariance of the model. The answer is that the model lost its gauge invariance and it can be shown performing the study of symmetries through computation of the gauge generator chains. For this purpose, a $2+1$ decomposition of (53) is performed, this means

$$\begin{aligned} S^{(3)L}{}_m = \kappa^2 \Big\langle [-\dot{k}_{0i} + 2\partial_i k_{00} - 2\partial_n k_{ni} - 2\epsilon_{in}\dot{v}_n + 2\epsilon_{in}\partial_n v_0]\dot{k}_{0i} + \dot{k}_{ij}\dot{k}_{ij} \\ +[2\epsilon_{nj}\partial_n k_{00} + 2\epsilon_{nj}\partial_m k_{nm} - \dot{v}_j - 2\partial_j v_0]\dot{v}_j + 2(\dot{v}_0)^2 + k_{00}\Delta k_{00} \\ -2k_{0i}\Delta k_{0i} + k_{ij}\Delta k_{ij} - (\partial_i k_{i0})^2 + \partial_n k_{ni}\partial_m k_{mi} - 2\epsilon_{ij}\partial_i v_j \partial_n k_{n0} \\ -2\epsilon_{lm}\partial_m v_0 \partial_n k_{nl} + v_0\Delta v_0 - v_i\Delta v_i + (\partial_n v_n)^2 \\ +m^2[2k_{0i}k_{0i} - k_{ij}k_{ij} - 2k_{00}k_{ii} + (k_{ii})^2]\Big\rangle \ , \end{aligned} \tag{84}$$

where $\epsilon_{ij} \equiv \epsilon^0{}_{ij}$ and $\Delta \equiv \partial_i\partial_i$.

Next, the momenta are

$$\Pi \equiv \frac{\partial \mathcal{L}}{\partial \dot{k}_{00}} = 0 \ , \tag{85}$$

$$\Pi^i \equiv \frac{\partial \mathcal{L}}{\partial \dot{k}_{0i}} = -2\dot{k}_{0i} - 2\epsilon_{in}\dot{v}_n + 2\partial_i k_{i0} - 2\partial_n k_{ni} + 2\epsilon_{in}\partial_n v_0 \ , \tag{86}$$

$$\Pi^{ij} \equiv \frac{\partial \mathcal{L}}{\partial \dot{k}_{ij}} = 2\dot{k}_{ij} \ , \tag{87}$$

$$P \equiv \frac{\partial \mathcal{L}}{\partial \dot{v}_0} = 4\dot{v}_0 \ , \tag{88}$$

$$P^j \equiv \frac{\partial \mathcal{L}}{\partial \dot{v}_j} = -2\epsilon_{nj}\dot{k}_{0n} - 2\dot{v}_j + 2\epsilon_{nj}\partial_n k_{00} + 2\epsilon_{nj}\partial_m k_{mn} - 2\partial_j v_0 \ , \tag{89}$$

and we establish the following commutation rules

$$\{k_{00}(x), \Pi(y)\} = \{v_0(x), P(y)\} = \delta^2(x-y) \ , \tag{90}$$

$$\{k_{0i}(x), \Pi^j(y)\} = \{v_i(x), P^j(y)\} = \delta^j{}_i\delta^2(x-y) \ , \tag{91}$$

$$\{k_{ij}(x), \Pi^{nm}(y)\} = \frac{1}{2}(\delta^n{}_i\delta^m{}_j + \delta^m{}_i\delta^n{}_j)\delta^2(x-y) \ . \tag{92}$$

It can be noted that (85) is a primary constraint that we name

$$G^{(K)} \equiv \Pi \ , \tag{93}$$

where K means the initial index corresponding to a possible gauge generator chain, provided by the algorithm developed in reference[7]. Moreover, manipulating (87) and (89), other primary constraints appear

$$G_i^{(K)} \equiv \partial_n k_{ni} - \epsilon_{in}\partial_n v_0 - \frac{\epsilon_{in}}{4}P^n + \frac{1}{4}\Pi^i \ , \tag{94}$$

and we observe that $G^{(K)}$ and $G_i^{(K)}$ are first class.

The preservation of constraints requires to obtain the Hamiltonian of the model. First at all, the Hamiltonian density can be written as $\mathcal{H}_0 = \Pi^i\dot{h}_{0i} + \Pi^{ij}\dot{h}_{ij} + P\dot{v}_0 + P^i\dot{v}_i - \mathcal{L}$, in other words

$$\begin{aligned}\mathcal{H}_0 = \frac{\Pi^{ij}\Pi^{ij}}{4} + \frac{P^2}{8} - \frac{P^iP^i}{4} + \epsilon_{nj}\partial_m k_{nm}P^j + v_0[\partial_i P^i + 4\epsilon_{ml}\partial_m\partial_n k_{nl}] \\ +k_{00}[2\partial_m\partial_n k_{mn} - \epsilon_{nm}\partial_n P^m + 2m^2 k_{ii}] + 2k_{0i}\Delta k_{0i} - k_{ij}\Delta k_{ij} \\ +(\partial_i k_{i0})^2 - 2\partial_n k_{ni}\partial_m k_{mi} + 2\epsilon_{ij}\partial_i v_j\partial_n k_{n0} + v_i\Delta v_i - (\partial_n v_n)^2 \\ -m^2[2k_{0i}k_{0i} - k_{ij}k_{ij} + (k_{ii})^2] \ .\end{aligned} \tag{95}$$

Then, the Hamiltonian is $H_0 = \int dy^2 \mathcal{H}_0(y) \equiv \left\langle \mathcal{H}_0 \right\rangle_y$ and the preservation of $G^{(K)}$, defined in (93) is

$$\{G^{(K)}(x), H_0\} = -2\partial_m\partial_n k_{mn}(x) + \epsilon_{nm}\partial_n P^m(x) - 2m^2 k_{ii}(x) \ . \tag{96}$$

The possible generators chain is given by the rule: "$G^{(K-1)}+\{G^{(K)}(x),H_0\}$ =*combination of primary constraints*", then

$$G^{(K-1)}(x) = 2\partial_m\partial_n k_{mn}(x) - \epsilon_{nm}\partial_n P^m(x) + 2m^2 k_{ii}(x)$$
$$+\big\langle a(x,y)G^{(K)}(y) + b^i(x,y)G_i^{(K)}(y)\big\rangle_y \ . \tag{97}$$

The preservation of $G_i^{(K)}$, defined in (94), is

$$\{G_i^{(K)}(x),H_0\} = \frac{\partial_n\Pi^{ni}(x)}{2} - \frac{\epsilon_{in}}{4}\partial_n P(x) + \frac{\epsilon_{in}}{2}\Delta v_n(x) + \frac{\epsilon_{in}}{2}\partial_n\partial_m v_m(x)$$
$$+\frac{\epsilon_{nm}}{2}\partial_i\partial_n v_m(x) - (\Delta - m^2)k_{0i}(x) \ , \tag{98}$$

then

$$G_i^{(K-1)}(x) = -\frac{\partial_n\Pi^{ni}(x)}{2} + \frac{\epsilon_{in}}{4}\partial_n P(x) - \frac{\epsilon_{in}}{2}\Delta v_n(x) - \frac{\epsilon_{in}}{2}\partial_n\partial_m v_m(x)$$
$$-\frac{\epsilon_{nm}}{2}\partial_i\partial_n v_m(x) + (\Delta - m^2)k_{0i}(x)$$
$$+\big\langle a^i(x,y)G^{(K)}(y) + b^i{}_j(x,y)G_j^{(K)}(y)\big\rangle_y \ . \tag{99}$$

The undefined objects $a(x,y)$, $b^i(x,y)$, $a^i(x,y)$ and $b^i{}_j(x,y)$ in expressions (97) and (99), are functions or distributions. If it is possible, they can be fixed in a way that the preservation of $G^{(K-1)}(x)$ and $G_i^{(K-1)}(x)$ would be combinations of primary constraints. With this, the generator chains could be interrupted and we simply take $K=1$. Of course, the order $K-1=0$ generators must be first class, as every one. Next we can see that all these statements depend on the massive or non-massive character of the theory.

Taking a chain with $K=1$, the candidates to generators of gauge transformation are (93), (94), (97) and (99). But, the only non null commutators are

$$\{G_i^{(1)}(x),G_j^{(0)}(y)\} = \frac{m^2}{4}\eta_{ij}\delta^2(x-y) \ , \tag{100}$$

$$\{G^{(0)}(x),G_i^{(0)}(y)\} = m^2\big(\partial_i\delta^2(x-y) + \frac{b^i(x,y)}{4}\big) \ , \tag{101}$$

saying that the system of "generators" is not first class. Moreover, the unsuccessful conditions (in the $m^2 \neq 0$ case) to interrupt the chains, are

$$\{G^{(0)}(x), H_0\} = m^2(\Pi^{nn}(x) - 2\partial_n k_{0n}(x)) \ , \tag{102}$$

$$\{G_i^{(0)}(x), H_0\} = m^2(\partial_n k_{in}(x) + \partial_i k_{00}(x) - \partial_i k_{nn}(x)) \ , \tag{103}$$

where we have fixed

$$a(x,y) = 0 \ , \tag{104}$$

$$b^i(x,y) = -2\partial^i \delta^2(x-y) \ , \tag{105}$$

$$a^i(x,y) = 0 \ , \tag{106}$$

$$b^i{}_j(x,y) = 0 \ . \tag{107}$$

All this indicates that in the case where $m^2 \neq 0$ there is not a first class consistent chain of generators and, then there is no gauge symmetry.

However, if we revisit the case $m^2 = 0$, conditions (102) and (103) are zero and the chains are interrupted. Now, the generators $G^{(1)}$, $G_i^{(1)}$, $G^{(0)}$ and $G_i^{(0)}$ are first class. Using (104)-(105), the generators are rewritten again

$$G^{(1)} \equiv \Pi \ , \tag{108}$$

$$G_i^{(1)} \equiv \partial_n k_{ni} - \epsilon_{in}\partial_n v_0 - \frac{\epsilon_{in}}{4}P^n + \frac{1}{4}\Pi^i \ , \tag{109}$$

$$G^{(0)} = -\frac{\epsilon_{nm}}{2}\partial_n P^m - \frac{\partial_n \Pi^n}{2} \ , \tag{110}$$

$$G_i^{(0)} = -\frac{\partial_n \Pi^{ni}}{2} + \frac{\epsilon_{in}}{4}\partial_n P - \frac{\epsilon_{in}}{2}\Delta v_n - \frac{\epsilon_{in}}{2}\partial_n\partial_m v_m - \frac{\epsilon_{nm}}{2}\partial_i\partial_n v_m + \Delta k_{0i} \ . \tag{111}$$

Introducing the parameters $\varepsilon(x)$ and $\varepsilon^i(x)$, a combination of (108)-(111) is taken into account in the way that the gauge generator is

$$G(\dot{\varepsilon},\dot{\varepsilon}^i,\varepsilon,\varepsilon^i)=\left\langle\dot{\varepsilon}(x)G^{(1)}(x)+\dot{\varepsilon}^i(x)G_i^{(1)}(x)+\varepsilon(x)G^{(0)}(x)+\varepsilon^i(x)G_i^{(0)}(x)\right\rangle\ , \quad (112)$$

and with this, for example the field transformation rules (this means, $\delta(...)=\{(...),G\}$) are written as

$$\delta k_{00}=\dot{\varepsilon}\ , \quad (113)$$

$$\delta k_{0i}=\frac{\dot{\varepsilon}^i}{4}+\frac{\partial_i\varepsilon}{2}\ , \quad (114)$$

$$\delta k_{ij}=\frac{1}{4}(\partial_i\varepsilon_j+\partial_j\varepsilon_i)\ , \quad (115)$$

$$\delta v_0=\frac{\epsilon_{nm}}{4}\partial_n\varepsilon_m\ , \quad (116)$$

$$\delta v_i=\frac{\epsilon_{in}}{4}\dot{\varepsilon}_n-\frac{\epsilon_{in}}{2}\partial_n\varepsilon\ , \quad (117)$$

and, redefining parameters as follows: $\varepsilon\equiv 2\xi_0$ and $\varepsilon^i=4\xi^i$, it is very easy to see that these rules match with (11) and (12), as we expected.

6 Concluding remarks

A perturbative regime based on arbitrary variations of the contorsion and metric as a (classical) fixed background, is performed in the context of a pure Yang-Mills formulation of the $GL(3,R)$ gauge group. There, we analyze in detail the physical content and the well known fact that a variational principle based on the propagation of torsion (contorsion), as dynamical and possible candidate for a quantum canonical description of gravity in a pure YM formulation get serious difficulties.

In the $2+1$ dimensional massless case we show that the theory propagates three massless degrees of freedom, one of them a non-unitary mode. Then, introducing appropiate quadratical terms dependent on torsion, which preserve parity and general covariance, we can see that the linearized limit do not reproduces an equivalent pure Hilbert-Einstein-Fierz-Pauli massive theory for a spin-2 mode and, moreover there is other non-unitary modes. Roughly speaking, at first sight one can blame it on the kinetic part of YM formulation because the existence of non-positive Hamiltonian connected with non-unitarity problem. Nevertheless there is other possible models of Gauss-Bonnet type which could solve the unitarity problem.

As we have given a glimpse, without introduction of explicit massive T^2-terms in the action, it is possible to breakdown the residual gauge invariance (i.e.: reduction of general covariance which survives after a perturbative procedure) as a consequence rising from the choice of a particular fixed non-Riemannian background. The question of the existence of the alleged geometrical mechanism which gives mass to the fields and the existence of extensions of YM model which cure non-unitarity(i.e.: a possible family of Gauss-Bonnet models with torsion), will be explored elsewhere.

Aknowlegments

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